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Note

Cubic graphs whose average number of regions is small[1]

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Abstract

Some previously investigated infinite families of cubic graphs have the property that the average number of regions of a randomly selected orientable embedding is proportional to the number of their vertices. This paper demonstrates that this property is not true of connected graphs in general. That is, for every sufficiently large even value of n, there is an n-vertex cubic graph G_n with fewer than $1+\ln(n+2)$ regions in its random orientable embedding. The proof provided is existential and no large cubic graphs are known that satisfy this scarceness of regions. It is conjectured that the complete graphs have a similar logarithmic bound and some numerical evidence is offered in support.

Two embeddings of a graph G on closed oriented surfaces are considered to be the *same* if at each vertex n of G, the surfaces' orientations induce the same cyclic permutation on the edges of G incident to n. In the terminology of [4], two embeddings are the same if and only if their rotation systems are identical. Subject to this definition, if the graph G has degree sequence $d_1, d_2, \ldots, d_n$, then it has a total of

$$\prod_{i=1}^{n} (d_i - 1)!$$

embeddings. Suppose this set of embeddings of G constitutes a sample space in which all the embeddings are assigned the same probability. Then $r_{\text{avg}}(G)$ denotes the expected number of regions in the randomly selected oriented embedding of G. In his paper [1] Dan Archdeacon asked whether, when G is restricted to connected cubic graphs, $r_{\text{avg}}(G)$ is roughly a linear function of the number of vertices of G. It follows

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from Corollary 2 of [9] that for such cubic graphs G

$$r_{\mathrm{avg}}(G) \leqslant n(2 + \ln 3)$$

and some infinite families of cubic graphs investigated in [3,5,8] also make such a linearity seem reasonable. Nevertheless, we demonstrate here that the answer to Archdeacon's question is negative.

The following vertex splitting operation is described in [2]. Let G be any simplicial (i.e., having no loops or multiple edges) graph and let u be any vertex of G of degree $d \geqslant 4$. If $u_1, u_2, \ldots, u_d$ are the vertices of G adjacent to u, then, for each $i = 2, 3, \ldots, d$, let G_i denote the graph obtained from G by deleting u and its incident edges and replacing them with the new vertices x and y and the new edges

$$[x,y],\ [x,u_1],\ [y,u_2],\ [y,u_3], \ldots, [y,u_{i-1}],\ [x,u_i],\ [y,u_{i+1}],\ [y,u_{i+2}], \ldots, [y,u_d].$$

For our purposes here it is necessary to extend this operation to nonsimplicial graphs as well. This is easily accomplished by subdividing the edges of the arbitrary graph G so as to obtain a simplicial graph G', applying the above splitting process to G' at u to obtain $G'_2, G'_3, \ldots, G'_d$, and reversing the subdivisions so as to obtain the desired $G_2, G_3, \ldots, G_d$. Figs. 1 and 2 contain an illustration of this process.

It is clear that the contraction of each G_i along the edge $[x,y]$ yields the original graph G. It is also clear that each such G_i has exactly one more edge and one more vertex than G, and that each G_i is connected if G is connected. As part of the proof of [2, Theorem 4.3], it is proven that when G is simplicial

$$\gamma_{\mathrm{avg}}(G) = \frac{1}{d-1} \sum_{i=2}^{d} \gamma_{\mathrm{avg}}(G_i), \tag{1}$$

where $\gamma_{\mathrm{avg}}(G)$ denotes the expected genus of the randomly selected embedding of G. The same proof works when G is not simplicial. Moreover, it follows from the Euler–Poincaré formula that

$$r_{\mathrm{avg}}(G) = \frac{1}{d-1} \sum_{i=2}^{d} r_{\mathrm{avg}}(G_i).$$

We shall refer to the G_i's as the *CG-descendants* of G. The *bouquet on n circles B_n* is the pseudograph that consists of n loops incident to a single vertex. It is known [11] that

$$\lim_{n \to \infty} [r_{\mathrm{avg}}(B_n) - H_{2n}] = 0, \tag{2}$$

where $H_m = \sum_{k=1}^{m} \frac{1}{k}$.

Proposition 1. *There exists an integer n_0 such that for each even integer $n > n_0$ there exists a connected cubic graph G_n such that G_n has n vertices and*

$$r_{\mathrm{avg}}(G_n) < 1 + \ln(n+2).$$

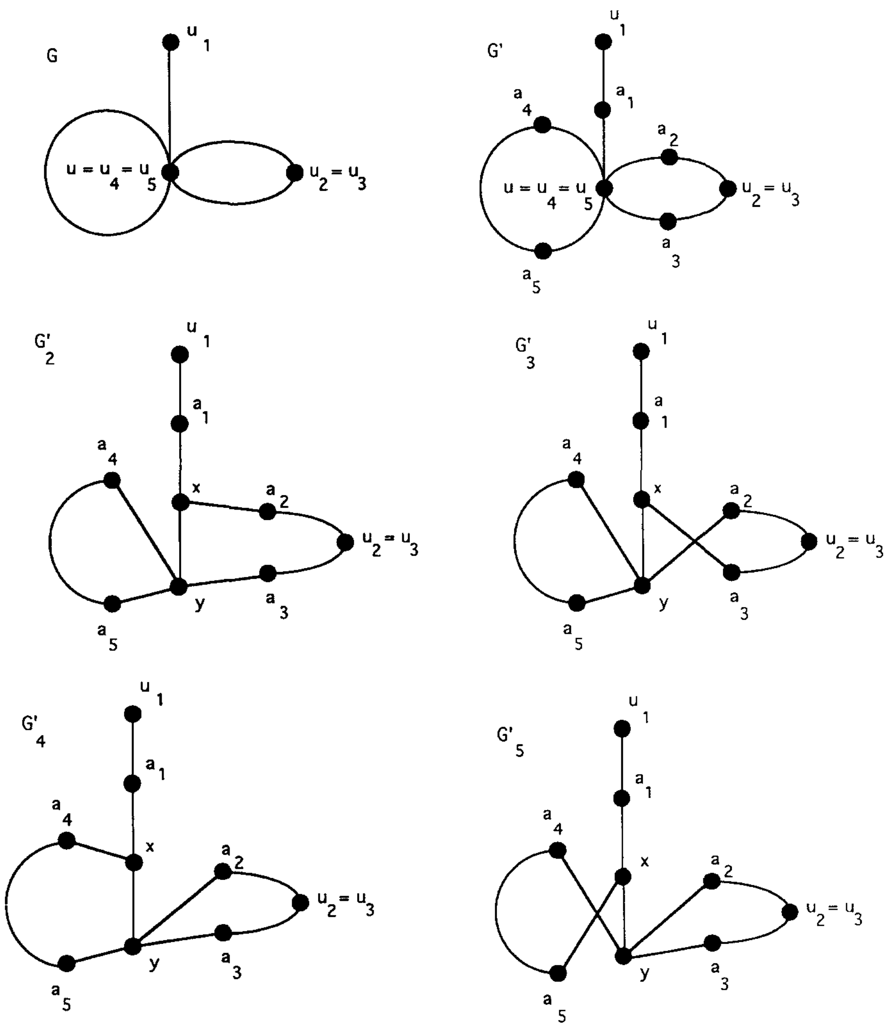

Fig. 1.

Proof. Let $\mathscr{L}_1$ denote the set consisting of all the CG-descendants of the bouquet of $\frac{1}{2}(n+2)$ circles $B_{(n+2)/2}$. Note that each graph of $\mathscr{L}_1$ has one vertex of degree 3 and one vertex of degree $n+1$. Assume that $\mathscr{L}_k$ has been defined for some positive integer $k \leqslant n-2$ and that each graph in $\mathscr{L}_k$ is connected, and has k vertices of degree 3 and one more vertex of degree $n+2-k$. Then $\mathscr{L}_{k+1}$ is the set of all the CG-descendants of all the graphs in $\mathscr{L}_k$ obtained by splitting at the unique vertex of degree greater than 3. Each graph in $\mathscr{L}_{k+1}$ is connected and contains $k+1$ vertices of degree 3 and one more vertex of degree $n+2-(k+1)$. The graphs of $\mathscr{L}_{n-1}$ are connected cubic graphs of order n, say $G_1, G_2, \ldots, G_s$. It follows from (1) that there exist weights $0 \leqslant \alpha_i \leqslant 1$,

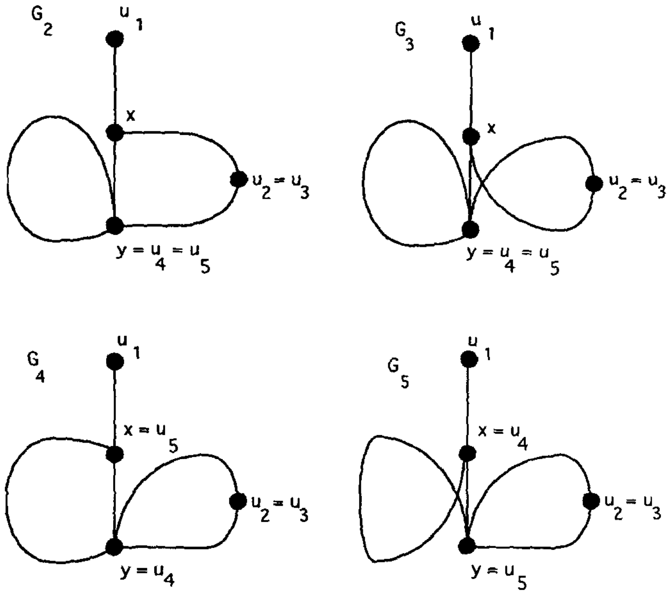

Fig. 2.

$i = 1, 2, \ldots, s$, such that

$$\sum_{i=1}^{s} \alpha_i = 1 \quad \text{and} \quad r_{\text{avg}}(B_{(n+2)/2}) = \sum_{i=1}^{s} \alpha_i r_{\text{avg}}(G_i). \tag{3}$$

It is well known that $\lim_{m\to\infty}(H_m - \ln m) = 0.5772\ldots$ (Euler's constant). Consequently, it follows from (2) that there exists an integer n_0 such that for even $n > n_0$ we have

$$|r_{\text{avg}}(B_{(n+2)/2}) - H_{n+2}| < \tfrac{1}{4} \quad \text{and} \quad |H_{n+2} - \ln(n+2)| < \tfrac{3}{4}.$$

Then, for $n > n_0$,

$$|r_{\text{avg}}(B_{(n+2)/2}) - \ln(n+2)| < 1.$$

Hence, it follows from (3) that for any fixed even $n > n_0$ there exists a graph G_n in $\mathscr{L}_{n-1}$ such that

$$r_{\text{avg}}(G_n) < 1 + \ln(n+2). \qquad \square$$

No large cubic graphs satisfying the conclusion of Proposition 1 are known. Nevertheless, the authors believe that most cubic graphs satisfy some slightly relaxed, but still logarithmic, version of Proposition 1's upper bound on the average number of regions. In fact, the authors believe that a similar bound holds for the complete graphs K_n as well. In response to a question of Richter's [6], the value of $r_{\text{avg}}(K_n)$ was estimated by means of random samples and the results are displayed in Fig. 3. For $n = 5, 6, 7, \ldots, 50$

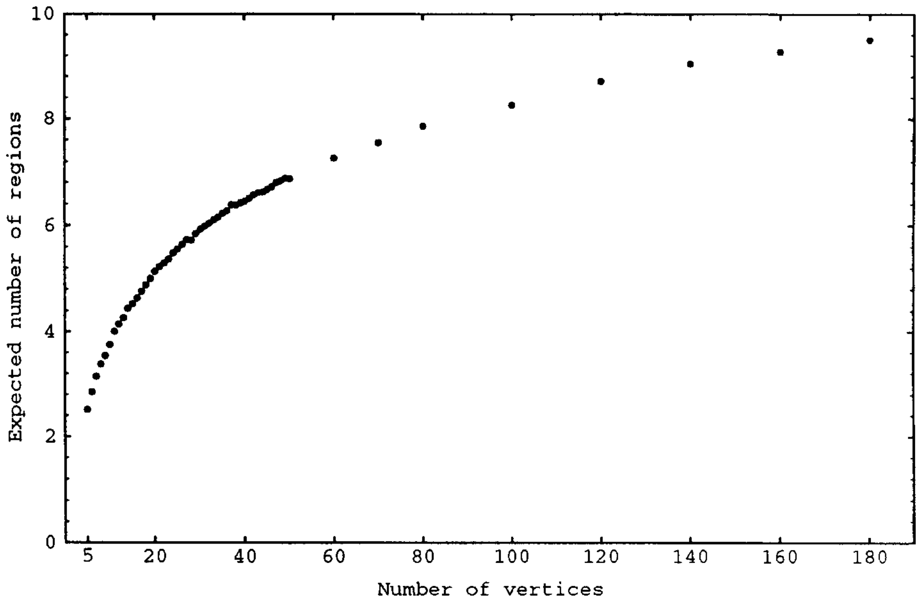

Fig. 3.

the sample sizes were 10 000 each, whereas for $n = 60, 70, 80, 100, 120, 140, 160, 180$ the sample size was 5000. Based on these experimental results we conjecture that

$$r_{\text{avg}}(K_n) = 2\ln(n) + \mathrm{O}(1).$$

So far, the best information regarding $r_{\text{avg}}(K_n)$ comes from [10] where it is proven that for each $\varepsilon > 0$ there exists a real number $b(\varepsilon)$ such that

$$r_{\text{avg}}(K_n) \leqslant (1+\varepsilon)(n-1) + b(\varepsilon).$$

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